

## Distinguishable Boxes

### Concepts

1. A lot of the cases with distinguishable boxes or dogs have already been covered in this class via our old methods of permutations and combinations. The new and more pertinent case that hasn't been covered is indistinguishable balls and distinguishable boxes. For this, if we have  $b$  identical biscuits and  $d$  distinguishable dogs, there are  $\binom{b+d-1}{b}$  ways to distribute the biscuits to the dogs.

### Examples

2. Suppose I am catering from Yali's and want to buy sandwiches to feed 60 students. How many ways can I do this if they have 8 sandwich options? How many ways can I do this if I want to get at least 2 of each sandwich?

**Solution:** We can think of putting indistinguishable balls into boxes, one box for each sandwich option. Thus, we need 8 boxes which is 7 dividers. Thus, there are  $\binom{67}{7}$  ways to do this.

If I want at least 2 of each sandwich, what I can do is buy two of each sandwich first, then I only need to buy 44 more sandwiches and I can do this  $\binom{44+7}{7} = \binom{51}{7}$  different ways.

3. How many ways can I distribute 30 course spots amongst 4 grades (freshmen, sophomores, juniors, seniors) so that there are no more than 10 freshmen in the course?

**Solution:** We already know how to do this if we require there to be more than 10 freshmen in the course because we just put 11 freshmen in the course and then do our regular distributing of indistinguishable balls into distinguishable boxes. Doing this, after putting 11 freshmen in, there are a total of  $\binom{19+4-1}{4-1} = \binom{22}{3}$  different ways. Thus, there are  $\binom{22}{3}$  ways for this to fail and in order to count the right number of ways, we can use complementary counting. There are  $\binom{30+4-1}{4-1} = \binom{33}{3}$  different ways to do this regularly. Therefore, there are  $\binom{33}{3} - \binom{22}{3}$  ways to do this in order for there to be at most 10 freshmen in the course.

## Problems

4. **TRUE** False In first example, since each student is getting a sandwich, the balls are sandwiches and the urns are students.

**Solution:** It is the opposite way around because we want to know the distribution of how many students get a particular type of sandwich, so how many balls are in a particular urn.

5. **TRUE** False The number of ways to distribute 8 identical balls to 12 urns is the same as the number of ways to distribute 11 identical biscuits to 9 dogs.

**Solution:** The first number is  $\binom{8+12-1}{8} = \binom{19}{8}$ . The second number is  $\binom{11+9-1}{11} = \binom{19}{11}$ . These two numbers are the same because  $\binom{n}{k} = \binom{n}{n-k}$ .

6. How many ways are there to deal hands of five cards to each of six players from a deck containing 52 different cards?

**Solution:** There are  $\binom{52}{5}$  ways to choose the cards for the first person. Then there are  $\binom{47}{5}$  ways to choose the cards for the second,  $\binom{42}{5}$  for the third and so on. This gives a total amount of

$$\binom{52}{5} \binom{47}{5} \binom{42}{5} \binom{37}{5} \binom{32}{5} \binom{27}{5}.$$

7. How many ways can you deal the 52 cards of a deck to 4 people so that everyone gets 13 cards and the oldest player gets the queen of spades?

**Solution:** We can assign the oldest player the queen of spades and then there are  $\binom{51}{12}$  ways to assign the rest of his/her cards. For the other people, there are  $\binom{39}{13}$ ,  $\binom{26}{13}$  and  $\binom{13}{13}$  ways. Thus there are a total of

$$\binom{51}{12} \binom{39}{13} \binom{26}{13} \binom{13}{13} = \binom{51}{12, 13, 13, 13}$$

ways.

8. You are buying bread for a soccer team. A bread shop has white bread, whole wheat, rye, sesame seed, pumpernickel, and gluten free. How many ways are there to choose 2 dozen loaves of bread?

**Solution:** We want to select 24 loaves of bread from 6 options. We can imagine this as putting each loaf into a box representing what option we chose for it. So there are 24 indistinguishable biscuits and 6 distinguishable dogs which gives an answer of  $\binom{24+6-1}{24} = \binom{29}{24} = \binom{29}{5}$ .

9. How many solutions are there to  $x_1 + x_2 + x_3 + x_4 + x_5 = 20$  if all are positive integers and  $x_3 \leq 3$ ?

**Solution:** First, we are distributing 20 balls into 5 boxes such that the third box has at most 3 balls and all the boxes have at least one ball. We can do this by first distributing one ball into each, so we have 15 left to distribute, and the third can have at most 2 more. We do this via complementary counting. There are a total of  $\binom{15+5-1}{5-1} = \binom{19}{4}$  ways and the ways that fail are if we put 3 more balls into the third to have 12 left to distribute in  $\binom{12+5-1}{5-1} = \binom{16}{4}$  ways. So there are a total of  $\binom{19}{4} - \binom{16}{4}$  ways.

10. How many 3 digit numbers have a sum of digits equal to 9?

**Solution:** This is asking how many ways can we have  $x_1 + x_2 + x_3 = 9$  if  $x_1 \geq 1$  because it has to be a 3 digit number. This is the same as  $x_1 + x_2 + x_3 = 8$  without any restrictions and this can be done  $\binom{8+3-1}{3-1} = \binom{10}{2}$  ways.

11. How many 7 digit decreasing numbers are there? One example is 9777650.

**Solution:** This is the number of ways to choose 7 digits out of 10 total with repetition. There are  $\binom{7+10-1}{10-1} = \binom{16}{9}$  ways to do this. But, if we choose all 0's, then this is not a 7 digit number so we have to get rid of that case and that is the only bad case. So, there are a total of  $\binom{16}{9} - 1$  total ways.

12. (Challenge) How many numbers less than 1,000,000 have the sum of their digits equal to 10?

**Solution:** This is asking how many ways can we have  $x_1 + x_2 + \cdots + x_6 = 10$ . There are  $\binom{10+6-1}{6-1} = \binom{15}{5}$  ways to do this. But 6 of those ways have  $x_i = 10$  which is not possible. Therefore, there are a total of  $\binom{15}{5} - 6$  ways.

## Indistinguishable Boxes

### Concepts

13. The value  $S(n, k)$  represents the number of ways we can distribute  $n$  distinguishable biscuits to  $k$  identical dogs or identical non-empty piles. The value  $p_k(n)$  represents the number of ways to partition  $n$  identical biscuits to  $k$  identical non-empty piles.

There is a recursive formula for the Stirling numbers (but not partition numbers) as  $S(n + 1, k) = kS(n, k) + S(n, k - 1)$ .

### Example

14. How many ways are there to split 28 distinct students up into at most 6 different groups if the groups are not numbered? What if the students are not distinct?

**Solution:** Here, the groups will be the urns and students the balls. There are distinct balls and indistinguishable urns. There can be 1, 2, 3, 4, 5 or 6 groups. There are going to be  $S(28, 1) + S(28, 2) + S(28, 3) + S(28, 4) + S(28, 5) + S(28, 6)$  ways to split the students up into at most 6 groups.

If the students are not distinct, then the balls are indistinguishable and this can be done in  $p_1(28) + p_2(28) + p_3(28) + p_4(28) + p_5(28) + p_6(28)$  ways.

### Problems

15. True    **FALSE** The only way to determine what  $S(5, 3)$  is to list out all the possibilities.

**Solution:** There is a recursive formula  $S(n + 1, k) = kS(n, k) + S(n, k - 1)$  that we can use to figure out what it is.

16. **TRUE**    False The only way to determine what  $p_3(5)$  is to list out all the possibilities.

**Solution:** This is the only way you know how to for now.

17. True **FALSE** In order to determine the number of ways to distribute 10 distinguishable items into 3 identical boxes so that each box has at least 2 items, we can place one item in each box and this problem reduces to the regular case of distributing 7 items in 3 identical boxes which is  $S(7, 3)$  ways.

**Solution:** We can only use this strategy if the items are indistinguishable so that it doesn't matter which items we put in the boxes.

18. **TRUE** False In order to determine the number of ways to distribute 10 identical items into 3 identical boxes so that each box has at least 2 items, we can place one item in each box and this problem reduces to the regular case of distributing 7 items in 3 identical boxes which is  $p_3(7)$  ways.
19. There are 14 students that want to break off into 3 non-empty study groups. How many ways can this occur?

**Solution:**  $S(14, 3)$ .

20. How many ways can 5 different employees be assigned to 3 identical offices when each office must have at least one person?

**Solution:** The employees are distinguishable and we are placing them into identical offices so we use Stirling numbers. This is  $S(5, 3)$ .

21. I want to store my 200 Yu-gi-oh cards in 4 different identical boxes. How many ways can I do this if some boxes are allowed to be empty?

**Solution:** You can do this  $S(200, 1) + S(200, 2) + S(200, 3) + S(200, 4)$  ways.

22. How many ways are there to split 15 identical marbles to 5 different non-empty groups?

**Solution:** There are  $p_5(15)$  different ways to do this.

23. How many ways are there of distributing 30 identical objects into 3 boxes if each box must have at least 5 items?

**Solution:** We can put 4 of each item in each box and then this reduces to the regular case. Thus, we need to distribute 18 objects into 3 boxes, which can be done  $p_3(18)$  ways.